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On reductive operator algebras

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In this paper, all Hilbert spaces are assumed to be separable.

The following conjectures are well-known in operator theory (see, e.g. [4]):

(I) *The invariant subspace conjecture:* Every operator on a Hilbert space of dimension larger than one has a (non-trivial, proper, closed) invariant subspace.

(R) *The reductive operator conjecture:* Every reductive operator is normal.

(TA) *The transitive algebra conjecture:* The only weakly closed transitive algebra on a Hilbert space is the whole algebra $B(H)$.

(RA) *The reductive algebra conjecture:* Every weakly closed reductive algebra is self-adjoint.

(H) *The hyperinvariant subspace conjecture:* Every operator other than a scalar has a hyperinvariant subspace.

(Recall that an operator T is *reductive* if every invariant subspace of T reduces T. A subspace is *hyperinvariant* for T if it is invariant for every operator commuting with T. An algebra $\mathscr{A}$ of operators is said to be *reductive* if every subspace which is invariant under all the operators in $\mathscr{A}$ reduces all the operators in $\mathscr{A}$.)

It is obvious that

$$\begin{array}{ccccc} (R) & \Rightarrow & (I) & & \\ \Uparrow & & \Uparrow & \nwarrow & \\ (RA) & \Rightarrow & (TA) & \Rightarrow & (H) \end{array}$$

DYER and PORCELLI [2] proved that $(R)\Leftrightarrow(I)$. In what follows, we consider other inter-relationships of these conjectures. First we introduce some notation.

For an operator T, we write $T^{(n)}$ for the direct sum of n copies of T, and for an algebra $\mathscr{A}$, we write $\mathscr{A}^{(n)}$ for $\{T^{(n)}: T\in\mathscr{A}\}$ and $M_n(\mathscr{A})$ for the $n\times n$ matrices with entries in $\mathscr{A}$. For an algebra $\mathscr{A}$, we write Lat $\mathscr{A}$ for the set of all

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subspaces which are invariant under all the operators in $\mathscr{A}$ and we write $\mathscr{A}'$ for the commutant of $\mathscr{A}$.

Note that (RA) is the strongest statement among the above conjectures. We divide it into the following two weaker statements:

(RA)$'$ If an algebra $\mathscr{A}$ is reductive, then $\mathscr{A}'$ is self-adjoint.

(RA)$''$ If a weakly closed algebra $\mathscr{A}$ is reductive and $I\in\mathscr{A}$, then $\mathscr{A}=\mathscr{A}''$. Obviously (RA)$\Leftrightarrow$((RA)$'$ & (RA$''$)).

Theorem 1. *Statement* (RA)$'$ *is equivalent to each of the following:*

(1) *If an algebra* $\mathscr{A}$ *is reductive, then so is* $\mathscr{A}'$.

(2) *If an algebra* $\mathscr{A}$ *is reductive and* $\mathscr{A}=\mathscr{A}''$, *then* $\mathscr{A}$ *is self-adjoint.*

To prove this, we need two lemmas. The first is quite well-known (e.g., see [4] Theorem 7.1).

Lemma a. *An operator* T *is in the weak closure of an albegra* $\mathscr{A}$ *if and only if*

$$\mathrm{Lat}\,\mathscr{A}^{(n)} \subseteqq \mathrm{Lat}\,T^{(n)}$$

for infinitely many positive integers n.

It follows from the above lemma that a weakly closed algebra is self-adjoint if and only if $\mathscr{A}^{(n)}$ is reductive for all n.

Lemma b. *If* $\mathscr{A}$ *is reductive, then so is* $\mathscr{A}''$.

Proof. Let $M\in\mathrm{Lat}\,\mathscr{A}''$. Then $M\in\mathrm{Lat}\,\mathscr{A}$ since $\mathscr{A}\subseteqq\mathscr{A}''$. As $\mathscr{A}$ is reductive, $P_M\in\mathscr{A}'$ where P_M is the projection associated with M. Hence $P_M\in(\mathscr{A}'')'$ $(=\mathscr{A}')$, i.e., M reduces $\mathscr{A}''$.

Proof of Theorem 1. Obviously (RA)$'\Rightarrow$(1) and (RA)$'\Rightarrow$(2). Now assume (2). Let $\mathscr{A}$ be a reductive algebra and $\mathscr{B}=\mathscr{A}''$. Then, by Lemma b, $\mathscr{B}$ is reductive and $\mathscr{B}=\mathscr{B}''$. By our assumption, $\mathscr{B}$ is self-adjoint. Hence $\mathscr{A}'=\mathscr{B}'$ is also self-adjoint. Thus (2)$\Rightarrow$(RA)$'$.

Assume (1) and let $\mathscr{A}$ be a reductive algebra. Then, for any positive integer n, $M_n(\mathscr{A})$ is also a reductive algebra. Hence $\mathscr{A}'^{(n)}=M_n(\mathscr{A})'$ is reductive for every n. Therefore, by the remark following Lemma a, $\mathscr{A}'$ is self-adjoint.

The conjecture (TA) can also be separated into weaker statements:

(TA)$'$ If an algebra is transitive, then $\mathscr{A}'$ consists of scalars.

(TA)$''$ If a weakly closed algebra is transitive, then $\mathscr{A}=\mathscr{A}''$.

Obviously (RA)$'\Rightarrow$(TA)$'$. (RA)$''\Rightarrow$(TA)$''$ and (TA)$\Leftrightarrow$((TA)$'$ & (TA)$''$). Note that (TA)$'$ is equivalent to the hyperinvariant subspace conjecture (H).

The following theorem is the main result of the present paper. It is a generalization of the following result in [1]: The hyperinvariant subspace conjecture is equivalent to the statement: if $\{T\}'$ is reductive, then $\{T\}'$ is self-adjoint. The proof is inspired by [5].

Theorem 2. *The hyperinvariant subspace conjecture* (H) *is equivalent to* (RA)$'$.

Proof. We have seen that (RA)$'\Rightarrow$(H). It remains to show that (TA)$'\Rightarrow$(RA)$'$. To prove this, we need some results from [1]. Let $\mathscr{A}$ be a reductive algebra. Take a maximal direct integral decomposition of $\mathscr{A}$:

$$\mathscr{A} \sim \int_Z^{\oplus} \mathscr{A}(z)\, dm(z).$$

By Theorem 4.1 in [1], $\mathscr{A}(z)$ is transitive a.e. (m). Let $T\in\mathscr{A}'$. We are going to show that $T^*\in\mathscr{A}'$.

For convenience, we call a finite collection $\mathscr{P}=\{P_1, P_2, \ldots, P_n\}$ of hermitian projections a *partition* if: (1) Each P_j is a diagonal operator with respect to the above decomposition, (2) $P_jP_k=0$ for $j\neq k$, and (3) $P_1+P_2+\ldots+P_n=I$. A partition $\mathscr{P}$ is a refinement of a partition $\mathscr{Q}$ and we write $\mathscr{P}\geqq\mathscr{Q}$ if for each P in $\mathscr{P}$ there is some Q in $\mathscr{Q}$ such that $PQ=P$. It is easy to see that there is a sequence of partitions $\{\mathscr{P}_n\}$ such that:

$$\mathscr{P}_1 \leqq \mathscr{P}_2 \leqq \mathscr{P}_3 \leqq \ldots$$

and the abelian von Neumann algebra generated by $\bigcup_{n=1}^{\infty}\mathscr{P}_n$ is the diagonal algebra $\mathscr{Z}$. Suppose $\mathscr{P}_n=\{P_{n,1}, P_{n,2}, \ldots, P_{n,m(n)}\}$. Put

$$T_n = \sum_{k=1}^{m(n)} P_{n,k}TP_{n,k}.$$

Note that, for $j\neq k$, $P_{n,j}TP_{n,k}$ is a nilpotent operator in $\mathscr{A}'$. Hence by [4] Lemma 9.2, $P_{n,j}T^*P_{n,k}\in\mathscr{A}'$. Therefore, $T^*-T_n^*\in\mathscr{A}'$ for each n.

Obviously $\|T_n\|\leqq\|T\|$ for each n. Hence $\{T_n\}_n$ has a subsequence, say $\{T_{n_k}\}_k$, which converges in the weak operator topology to S, say. It is easy to see that $SP=PS$ for each $P\in\bigcup_{n=1}^{\infty}\mathscr{P}_n$. Hence $S\in\mathscr{Z}'$. Therefore, S is decomposable, say $S=\int_Z^{\oplus} S(z)\, dm(z)$. Since $T_{n_k}\in\mathscr{A}'$ for each k, we also have $S\in\mathscr{A}'$. Therefore $S(z)\in\mathscr{A}(z)'$ a.e. (m). Since $\mathscr{A}(z)$ is transitive a.e. (m), by our assumption (TA)$'$, $S(z)$ is a scalar. Therefore S is a normal operator in $\mathscr{A}'$. By Fuglede's theorem, $S^*\in\mathscr{A}'$.

Since $T^*-T^*_{n_k}\in\mathscr{A}'$ for each k, we have $T^*-S^*\in\mathscr{A}'$. Hence $T^*=(T^*-S^*)+$ $+S^*\in\mathscr{A}'$.

By using the same argument, we can show:

Theorem 3. *If $\mathscr{A}$ is a reductive algebra and $\mathscr{A}\sim\int_Z^{\oplus}\mathscr{A}(z)\,dm(z)$ is a direct integral decomposition of $\mathscr{A}$ such that $\mathscr{A}(z)'$ is self-adjoint a.e. (m), then $\mathscr{A}'$ is self-adjoint.*

Corollary. *If $\mathscr{Z}$ is an abelian von Neumann algebra, n a positive integer and $\mathscr{A}$ a reductive algebra contained in $M_n(\mathscr{Z})$, then $\mathscr{A}'$ is self-adjoint.*

Proof. There is a finite measure space (Z, m) such that $\mathscr{Z}$ corresponds to multiplication operators acting on $L^2(Z, m)$. (See, for example, Radjavi and Rosenthal [4] p. 124.) Let K be an n-dimensional Hilbert space and $z\to H(z)$ be the constant field of Hilbert spaces with $H(z)\equiv K$. Then $\mathscr{A}$ becomes a reductive algebra consisting of decomposable operators. We write

$$\mathscr{A}\sim\int_Z^{\oplus}\mathscr{A}(z)\,dm(z).$$

By Theorem 4.1 in [1], $\mathscr{A}(z)$ is reductive a.e. (m). Since $\mathscr{A}(z)\subseteqq B(K)$ and dim $K=$ $=n<\infty$, $\mathscr{A}(z)$ is self-adjoint a.e. (m). Now the corollary follows from Theorem 3.

Let $\mathscr{A}$ be a reductive algebra. The von Neumann algebra $\mathscr{I}(\mathscr{A})$ generated by $\{P_M: M\in\operatorname{Lat}\mathscr{A}\}$ is called the *invariant algebra* of $\mathscr{A}$ and was introduced by Hoover [3]. Let $\mathscr{Z}(\mathscr{A})$ be the centre of $\mathscr{I}(\mathscr{A})$. Then the reductive algebra conjecture (RA) can be rendered into the following two weaker statements:

(RA1) If $\mathscr{A}$ is a reductive algebra, and $\mathscr{Z}(\mathscr{A})\subseteqq\mathscr{A}$, then $\mathscr{A}$ is self-adjoint.

(RA2) If $\mathscr{A}$ is a reductive algebra, then $\mathscr{Z}(\mathscr{A})\subseteqq\mathscr{A}$.

Obviously (RA)$\Leftrightarrow$((RA1) & (RA2)).

Since, for a reductive algebra $\mathscr{A}$, $\mathscr{Z}(\mathscr{A})\subseteqq\mathscr{A}''$ (see e.g. [3] Corollary 1), we have (RA)$''\Rightarrow$(RA2). By the same reasoning it follows from Theorem 1 that (RA1)$\Rightarrow$(RA)$'$.

In [1], it was proved that the following two statements are equivalent:

(CTA) An abelian algebra on a Hilbert space is intransitive.

(CRA) If $\mathscr{A}$ is an abelian reductive algebra, then $\mathscr{A}$ is self-adjoint.

Remark. If $\{\mathscr{A}_j\}$ is a collection of reductive algebras, then the weakly closed algebra generated by $\bigcup_j\mathscr{A}_j$ is also reductive. Thus, by Zorn's lemma, every abelian reductive algebra is contained in a maximal abelian reductive algebra.

Theorem 4. *If* $\mathscr{A}$ *is a maximal abelian reductive algebra, then* $\mathscr{A}=\mathscr{A}'$.

Proof. Suppose the contrary. Then there is an operator T in $\mathscr{A}'$ which is not in $\mathscr{A}$. Let $\mathscr{B}$ be the algebra generated by $\mathscr{A}$ and T. Then $\mathscr{B}$ is an abelian algebra properly containing $\mathscr{A}$. By the first sentence of the above remark, $\mathscr{A}$ contains all projections in $\mathscr{A}'$. Let P be a projection onto an invariant subspace of $\mathscr{B}$. Then $P\in\mathscr{A}'$ since $\mathscr{A}\subseteq\mathscr{B}$ and $\mathscr{A}$ is reductive. Hence $P\in\mathscr{A}$. As $T\in\mathscr{A}'$, we have $TP=PT$. Therefore $P\in\mathscr{B}'$. We see that $\mathscr{B}$ is also an abelian reductive algebra. Contradiction.

References

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